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Considering heterogeneity in a cylindrical section of a tree

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Abstract

Published values for the elastic coefficients of wood indicate that this material may be considered orthotropic with respect to the cylindrical coordinates. This indicates that simplifying the elasticity tensor to allow for the non-unique strains at $r = 0$, is a simplification that may ignore important structural characteristics of a tree. The constitutive equations for a cylindrical section of a tree were posed in cylindrical coordinates as a linear function of the radial coordinate r. The constitutive equations were transformed to a Cartesian basis so that a solution to Saint-Venant's Problem, proposed by Iesan (Lecture Notes in Mathematics (1987) 161) could be employed for a cylindrical section of a tree. From Iesan's solution it was possible to determine that the auxiliary generalized plane strain stresses can only be a function of r, and that the total stresses (in cylindrical coordinates) in the plane of the transverse cross-section must be equal to zero.

Keywords: Wood; Heterogeneous; Elasticity; Anisotropy; Orthotropic; Beam

1. Introduction

Lyons et al. (2002) noted that wood in the bole of a tree might be considered a linear elastic material that is orthotropic with respect to the cylindrical coordinate axes (Fig. 1). If the basis defining the constitutive equations is transformed by a clockwise rotation to the Cartesian frame, then a solution to the Relaxed Saint-Venant's Problem, proposed by Iesan (1987), may be used to determine the stresses and displacements for a cylindrical section of a tree.

Lyons et al. (2002) considered the problem where the elastic coefficients were constant in a cylindrical section of a tree. This placed certain constraints on the elastic coefficients that resulted in the material acting similarly to a transversely isotropic material when used in the Relaxed Saint-Venant's Problem. The Wood handbook (USDA, 1974) presents values for the elastic coefficients of *Pseudotsuga menziesii* (Douglas fir) that indicate the simplified coefficient matrix required to allow for the non-unique strains at $r = 0$, is a simplification that may ignore important structural characteristics of a tree.

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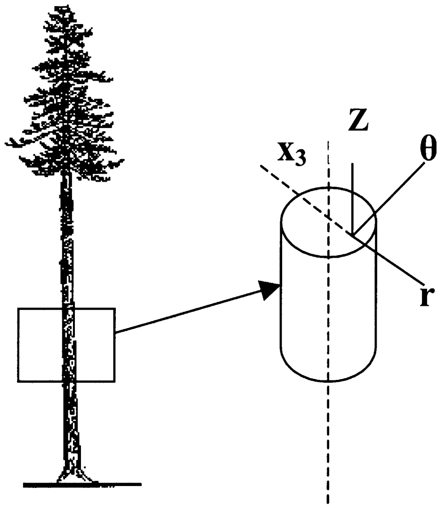

Fig. 1. Axes of anisotropy in a cylindrical section of a tree.

The objective of this paper is to propose a constitutive equation for a cylindrical section of a tree that allows consideration of the orthotropic material properties and heterogeneity of wood in trees. Considering this constitutive equation, and the solution to the Relaxed Saint-Venant's Problem proposed by Iesan (1987), we will prove that the stresses from the auxiliary generalized plane strain problems are functions of the cylindrical coordinate r alone. In addition, we will prove that the two normal stresses $(S_{rr}, S_{\theta\theta})$ and the shear stress $(S_{r\theta})$ in the plane of the transverse cross-section of a tree are equal to zero.

2. Constitutive equations for the bole of a tree

The constitutive equations for a linear elastic material that is orthotropic in cylindrical coordinates are (prime denotes basis in cylindrical coordinates)

$$\begin{aligned} S'_{ij} &= C'_{ijkl}E'_{kl} \\ E'_{ij} &= S'_{ijkl}S'_{kl} \end{aligned} \tag{2.1}$$

where S'_{ij} is Cauchy's stress tensor, E'_{ij} is the infinitesimal strain tensor, C'_{ijkl} is the elasticity tensor, and S'_{ijkl} is the compliance tensor.

The Wood handbook (USDA, 1974) presents the engineering constants for Douglas fir when the wood is assumed to be orthotropic in cylindrical coordinates (Table 1). The coefficients from the upper triangle of the compliance matrix, calculated from the values in Table 1, are presented in Table 2.

Consider a cylindrical section of a tree that is orthotropic with z, r, and θ being the axes of anisotropy. If the z-axis falls within the cylindrical section of the tree then certain relations are required between the elastic coefficients (Lekhnitskii, 1981). In cylindrical coordinates when $r = 0$, the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ become indistinguishable. Therefore, it must be possible to interchange the r and θ directions in (2.1); this requires certain of the coefficients to be equal at $r = 0$

$$\left.\begin{matrix} S'_{1111} = S'_{2222} & S'_{1133} = S'_{2233} & S'_{2332} = S'_{1313} \\ C'_{1111} = C'_{2222} & C'_{1133} = C'_{2233} & C'_{2332} = C'_{1313} \end{matrix}\right\} \quad \text{at } r = 0 \tag{2.2}$$

Table 1
Engineering constants for Douglas fir (USDA, 1974)

E_Z[a] (Pa)	E_θ (Pa)	E_R (Pa)	$G_{\theta Z}$[b] (Pa)	G_{RZ} (Pa)	$G_{\theta R}$ (Pa)	v_{ZR}[c]	$v_{Z\theta}$	$v_{R\theta}$	v_{RZ}	$v_{\theta R}$	$v_{\theta Z}$
1.08E + 10	5.40E + 08	7.34E + 08	6.91E + 08	8.42E + 08	7.56E + 07	0.292	0.449	0.390	0.287	0.020	0.022

[a] Young's modulus.
[b] Shear modulus.
[c] Poisson's ratio.

Table 2
Compliance coefficients for Douglas fir

S'_{1111} (Pa^{-1})	S'_{1122} (Pa^{-1})	S'_{1133} (Pa^{-1})	S'_{2222} (Pa^{-1})	S'_{2233} (Pa^{-1})	S'_{3333} (Pa^{-1})	S'_{2323} (Pa^{-1})	S'_{1313} (Pa^{-1})	S'_{1212} (Pa^{-1})
1.36E − 09	−3.70E − 10	−2.70E − 11	1.85E − 09	−4.16E − 11	9.26E − 11	1.45E − 09	1.19E − 09	1.32E − 08

However, the compliance coefficients (Table 2) indicate that Eq. (2.2) is not true for all points in the cross-section of a tree. It is necessary to introduce constitutive equations that are a function of the cylindrical coordinate r in order to satisfy (2.2) at $r = 0$, while allowing for other combinations of coefficients where $r \neq 0$. The following constitutive equation in cylindrical coordinates will be used to model a cylindrical section of a tree,

$$\left.\begin{aligned} E'_{ij} &= S'_{ijkl}S'_{kl} = \lfloor \underline{S_{ijkl}} + r^*\underline{M_{ijkl}} \rfloor S'_{kl} \\ S'_{ij} &= C'_{ijkl}E'_{kl} = [\underline{C_{ijkl}} + r^*\underline{K_{ijkl}}]E'_{kl} \end{aligned}\right\} \quad \text{in cylindrical coordinates} \tag{2.3}$$

where $\underline{S_{ijkl}}$, $\underline{M_{ijkl}}$, $\underline{C_{ijkl}}$, and $\underline{K_{ijkl}}$ are constants.

Lyons et al. (2002) transformed the elasticity and compliance tensors from a cylindrical basis to a Cartesian basis,

$$\begin{aligned} C_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}C'_{mnrs} \\ S_{ijkl} &= Q_{mi}Q_{nj}Q_{rk}Q_{sl}S'_{mnrs} \end{aligned} \tag{2.4}$$

where $Q_{ij} = Q_{ij}(\theta)$ is a clockwise rotation about the x_3-axis. The complete list of transformation equations is included in Appendix A. Given (2.4) the constitutive equations can be written in Cartesian coordinates

$$S_{ij} = C_{ijkl}E_{kl} \quad E_{ij} = S_{ijkl}S_{kl} \tag{2.5}$$

Before transforming the compliance and elasticity coefficients in (2.3) to Cartesian coordinates, some simplifications can be made. Eq. (2.2) does not place any restrictions on S'_{1122}, S'_{3333}, S'_{1212}, or C'_{1122}, C'_{3333}, C'_{1212}. Therefore, these coefficients may be independent of r and so the following simplifications can be made. Let

$$\begin{aligned} \underline{K_{1122}} &= \underline{K_{3333}} = \underline{K_{1212}} = 0 \\ \underline{M_{1122}} &= \underline{M_{3333}} = \underline{M_{1212}} = 0 \end{aligned} \tag{2.6}$$

Eq. (2.2) does place restrictions on (2.3) when $r = 0$, therefore, let

$$\left.\begin{matrix} \underline{C_{2222}} = \underline{C_{1111}} & \underline{C_{2233}} = \underline{C_{1133}} & \underline{C_{2323}} = \underline{C_{1313}} \\ \underline{S_{2222}} = \underline{S_{1111}} & \underline{S_{2233}} = \underline{S_{1133}} & \underline{S_{2323}} = \underline{S_{1313}} \end{matrix}\right\} \quad \text{at } r = 0 \tag{2.7}$$

Substitute (2.6) into (2.3) when $r = 0$, substitute this into the fourth and fifth equations of (A.1), then the following can be noted,

$$\left.\begin{aligned} C_{2323} &= S_\theta^2 \underline{C_{1313}} + C_\theta^2 \underline{C_{2323}} = [S_\theta^2 + C_\theta^2]\underline{C_{1313}} = \underline{C_{1313}} \\ C_{1313} &= C_\theta^2 \underline{C_{1313}} + S_\theta^2 \underline{C_{2323}} = [C_\theta^2 + S_\theta^2]\underline{C_{1313}} = \underline{C_{1313}} \\ S_{2323} &= S_\theta^2 \underline{S_{1313}} + C_\theta^2 \underline{S_{2323}} = [S_\theta^2 + C_\theta^2]\underline{S_{1313}} = \underline{S_{1313}} \\ S_{1313} &= C_\theta^2 \underline{S_{1313}} + S_\theta^2 \underline{S_{2323}} = [C_\theta^2 + S_\theta^2]\underline{S_{1313}} = \underline{S_{1313}} \end{aligned}\right\} \text{ at } r = 0 \tag{2.8}$$

To form the compliance coefficients in Cartesian coordinates substitute (2.3) into (A.1), then take into account (2.6)–(2.8). The resulting compliance coefficients in Cartesian coordinates may be found in Appendix B.

3. Elastic equations

With the compliance coefficients transformed into Cartesian coordinates, Eq. (B.1), it is possible to use Iesan's (1987) formulation to solve the Relaxed Saint-Venant's Problem. Lyons et al. (2002) considered a cylindrical section of a tree as a Relaxed Saint-Venant's Problem with loads independent of x_3. The statement of the Relaxed Saint-Venant's Problem will be repeated here for convenience.

From now on in this paper, Greek indices will range from 1 to 2, while Latin indices range from 1 to 3 unless otherwise specified. Summation notation is used for repeated indices, and a comma followed by a subscript will indicate a partial derivative with respect to the coordinate. Note the following special notation will be used; the Kronecker delta function (δ_{ij}), and the two-dimensional alternator symbol ($e_{\alpha\beta}$).

Consider a cylindrical section of a tree as a cantilever beam with constant cross-sections (Fig. 2). Let Σ_1 be the open cross-section at $x_3 = 0$, let Σ_2 be the open cross-section at $x_3 = h$, and let Σ be an arbitrary open cross-section with normal x_3. The lateral surface of the cylinder will be Π, while the boundary of an arbitrary cross-section is Γ. The closure of an arbitrary cross-section will be $\overline{\Sigma} = \Sigma \cup \Gamma$.

The resultant loads applied to the cross-section at $x_3 = 0$ are the forces $\mathbf{F}$ and the moments $\mathbf{M}$, and these are represented by integral functions of the displacement vector $\mathbf{u}$, where $f(\mathbf{u}) = \mathbf{F}$ and $m(\mathbf{u}) = \mathbf{M}$. The lateral surface of the cylinder is unloaded, the cross-section at Σ_2 is fixed, and body loads will be ignored in

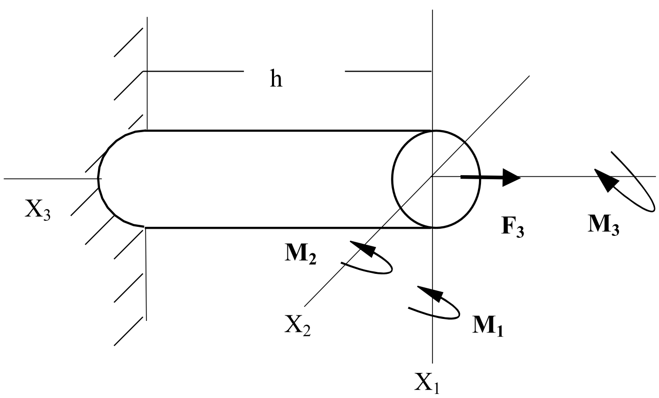

Fig. 2. Cylindrical cantilever beam subject to loads independent of x_3.

this analysis. The problem in Fig. 2 is of the class P_1 as defined by Iesan (1987), where the resultant loads acting on Σ are independent of x_3 and $F_\alpha = 0$.

The displacements resulting from strain, derived in a manner similar to that used by Iesan (1987), are

$$u_i = \delta_{i\alpha}\left[-a_\alpha \frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_\beta x_3 \right] + \delta_{i3}[a_\rho x_\rho + a_3]x_3 + W_i \tag{3.1}$$

where a_p are constants that will have to be determined using the boundary conditions, and $\mathbf{W} = \mathbf{W}(x_1, x_2)$ is a vector composed of the functions of integration.

Since the body forces are being ignored and the lateral surface of the cylinder is unloaded, the necessary conditions for a solution imply that the sum of the stress fields acting on Σ_2 must be in equilibrium with the resultant loads acting on Σ_1, therefore,

$$\begin{array}{ll} \int_{\Sigma_2} S_{\alpha 3}(\mathbf{u})\mathrm{d}a = -f_\alpha(\mathbf{u}) = 0 & \int_{\Sigma_2} S_{33}(\mathbf{u})\mathrm{d}a = -f_3(\mathbf{u}) = -F_3 \\ \int_{\Sigma_2} e_{\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u})\mathrm{d}a = -m_3(\mathbf{u}) = -M_3 & \int_{\Sigma_2} x_\alpha S_{33}(\mathbf{u})\mathrm{d}a = e_{\alpha\beta} m_\beta(\mathbf{u}) = e_{\alpha\beta} M_\beta \end{array} \tag{3.2}$$

Substituting (3.1) into the definition of the infinitesimal strain tensor, the resulting strains are

$$\begin{array}{lll} E_{11}(\mathbf{u}) = W_{1,1} & E_{22}(\mathbf{u}) = W_{2,2} & E_{33}(\mathbf{u}) = (a_\rho x_\rho + a_3) \\ E_{23}(\mathbf{u}) = \frac{1}{2}[a_4 x_1 + W_{3,2}] & E_{13}(\mathbf{u}) = \frac{1}{2}[-a_4 x_2 + W_{3,1}] & E_{12}(\mathbf{u}) = \frac{1}{2}[W_{1,2} + W_{2,1}] \end{array} \tag{3.3}$$

Consider the constitutive Eq. (2.5). Substitute the strain tensor (3.3) into the constitutive equation, then the stress tensor in Cartesian coordinates becomes

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W}) \tag{3.4}$$

The $T_{ij}(\mathbf{W}) = C_{ijk\alpha} W_{k,\alpha}$, are the stresses resulting from the displacement vector $\mathbf{W}$, which is independent of x_3 and so forms a generalized plain strain problem. Iesan (1987) found that the generalized plane strain problem could be separated into four auxiliary problems $T_{ij}^{(p)}$ $(p = 1, 2, 3, 4)$, which are defined by the following equilibrium Eq. (3.5a) and boundary conditions Eq. (3.5b).

$$T_{i\alpha}^{(\beta)}(\mathbf{W})_{,\alpha} + (C_{i\alpha 33} x_\beta)_{,\alpha} = 0 \quad T_{i\alpha}^{(3)}(\mathbf{W})_{,\alpha} + (C_{i\alpha 33})_{,\alpha} = 0 \quad T_{i\alpha}^{(4)}(\mathbf{W})_{,\alpha} - e_{\rho\beta}(C_{i\alpha\rho 3} x_\beta)_{,\alpha} = 0 \tag{3.5a}$$

$$T_{i\alpha}^{(\beta)}(\mathbf{W}) n_\alpha = -C_{i\alpha 33} x_\beta n_\alpha \quad T_{i\alpha}^{(3)}(\mathbf{W}) n_\alpha = -C_{i\alpha 33} n_\alpha \quad T_{i\alpha}^{(4)}(\mathbf{W}) n_\alpha = e_{\rho\beta} C_{i\alpha\rho 3} x_\beta n_\alpha \tag{3.5b}$$

Here $\mathbf{n}$ is the unit normal to Γ. The auxiliary problems combine as follows,

$$T_{ij}(\mathbf{W}) = \sum_{p=1}^{4} a_p T_{ij}^{(p)}(\mathbf{W}) \tag{3.6}$$

4. Dependence of the auxiliary generalized plane strain stresses on r

The solutions to the Relaxed Saint-Venant's Problem proposed by Iesan (1987), and Lekhnitskii (1981) both identify a generalized plane strain component of the problem. In Iesan's (1987) solution the generalized plane strain problem is further separated into four auxiliary generalized plane strain problems. When considering the constitutive equations (2.3) with loads independent of x_3, it is possible to see that the auxiliary generalized plane strain stresses $T_{ij}^{(p)}$ $p = (1, 2, 3, 4)$ are functions of r alone.

Theorem 1. *If* $T_{ij}^{(p)} = N_{ij}^{(p)}$ *(for all* i, j *except* $i = j = 3$*) at* $\mathbf{x} = \mathbf{x}(m_1, m_2, m_3) \in \Sigma$, *then* $T_{ij}^{(p)} = N_{ij}^{(p)}$ *on the closed circular path* $x_1^2 + x_2^2 = b^2$, *where* $N_{ij}^{(p)}$ *are constants except for* $N_{33}^{(p)}$, $m_1^2 + m_2^2 = b^2$, $0 \leqslant m_3 \leqslant h$, *and* $b \neq 0$.

Proof. Consider the original Cartesian coordinate system $\mathbf{x}$ (Fig. 2). A second Cartesian coordinate system can be formed by a counterclockwise rotation about the x_3-axis.

$$x_i^{\mathrm{II}} = Q_{ij}x_j \tag{4.1}$$

Here γ is a positive angle such that $0 < \gamma \leqslant 2\pi$, and

$$Q_{ij} = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the problem was originally defined the cross-section Σ was assumed circular, and the constitutive equations in the original cylindrical coordinates were independent of θ. Therefore, only the resultant loads, $\mathbf{F}$ and $\mathbf{M}$, are dependent on the original placement of the x_1-axis.

Consider Eqs. (3.5a) and (3.5b), which defines the auxiliary generalized plane strain stresses $T_{ij}^{(p)}$ for all i, j except $i = j = 3$. Let $d_j = d_j^{\mathrm{II}}$, where d_j is a point in the x_j frame, and d_j^{II} is a point in the x_j^{II} frame. Note the necessary conditions for a solution, Eq. (3.2), must be used to solve for the constants a_p in order to determine the generalized plane strain stresses in (3.6). Since the resultant loads occur in (3.2) the a_p are dependent on the $\mathbf{x}$ frame. However, the constants a_p do not occur in the auxiliary generalized plane strain problems, Eqs. (3.5a) and (3.5b), and so

$$T_{ij}^{(p)}(d_j) = T_{ij}^{(p)}(d_j^{\mathrm{II}}) \tag{4.2}$$

However, by the coordinate transformation (4.1), when $d_j = d_j^{\mathrm{II}}$, then

$$d_j^{\mathrm{II}} \neq Q_{ij}d_j \tag{4.3}$$

Therefore, the $T_{ij}^{(p)}$ (for all i, j except $i = j = 3$) must be invariant under the coordinate transformation (4.1) and so they must be constant along the circular path $x_1^2 + x_2^2 = b^2$. Thus, Theorem 1 is proven for all $\mathbf{x} \in \Sigma$ except at $b = 0$. □

Corollary 1. *If $T_{ij}^{(p)} = N_{ij}^{(p)}$ (for all i, j except $i = j = 3$) at $\mathbf{x} = \mathbf{x}(m_1, m_2, m_3) \in \Sigma$, then $T_{33}^{(p)} = N_{33}^{(p)}$ on the closed circular path $x_1^2 + x_2^2 = b^2$, where $N_{33}^{(p)}$ is a constant, $m_1^2 + m_2^2 = b^2$, $0 \leqslant m_3 \leqslant h$, and $b \neq 0$.*

Proof. Recall the definition of the infinitesimal strains and that the generalized plane strain displacements are $\mathbf{W} = \mathbf{W}(x_1x_2)$, then

$$E_{33}^{(p)}(\mathbf{W}) = W_{3,3}^{(p)} = 0 \tag{4.4}$$

Consider the constitutive equation (2.5) in the auxiliary generalized plane strain problem for $i = j = 3$, and note (4.4), then

$$E_{33}^{(p)}(\mathbf{W}) = S_{3311}T_{11}^{(p)} + S_{3322}T_{22}^{(p)} + S_{3333}T_{33}^{(p)} + 2S_{3312}T_{12}^{(p)} = 0 \tag{4.5}$$

Solve (4.5) for $T_{33}^{(p)}$, then

$$T_{33}^{(p)} = -\frac{1}{S_{3333}}(S_{3311}T_{11}^{(p)} + S_{3322}T_{22}^{(p)} + 2S_{3312}T_{12}^{(p)}) \tag{4.6}$$

By Theorem 1 the $T_{\alpha\beta}^{(p)}$ in (4.6) are functions of r alone. The coefficients $S_{\alpha\beta 33}$ are functions of r and θ by Eqs. (2.3) and (2.4). Let θ^{II} be the cylindrical coordinate measured off the x_1^{II}-axis, and let θ be the cylindrical coordinate measured off the x_1-axis. Let $\theta^{\mathrm{II}} = \theta$, then

$$S_{\alpha\beta 33}(\theta^{\mathrm{II}}, r) = S_{\alpha\beta 33}(\theta, r) \tag{4.7}$$

and

$$T_{33}^{(p)}(\theta^{\mathrm{II}}, r) = T_{33}^{(p)}(\theta, r) \tag{4.8}$$

However, by the transformation (4.1)

$$\theta^{\mathrm{II}} = \theta + \gamma \tag{4.9}$$

Therefore, by (4.8) $T_{33}^{(p)}$ must be invariant under the transformation (4.1) and so must be constant on the circle $x_1^2 + x_2^2 = b^2$ and Corollary 1 is proven. □

5. The auxiliary generalized plane strain stresses as potential functions

Recall the first equation of (3.5a) and let $i = 1$, $\alpha = 1, 2$, and $\beta = 1$, then

$$T_{11}^{(1)}(\mathbf{W})_{,1} + T_{12}^{(1)}(\mathbf{W})_{,2} = -(C_{1133}x_1)_{,1} - (C_{1233}x_1)_{,2} \tag{5.1}$$

Note from (B.1) that the $C_{i\alpha 33}$ are functions of r and θ, therefore, the right hand side of (5.1) can be written as a function of r and θ. Let

$$C_{1133}x_1 = D_{11}(r, \theta) \quad C_{1233}x_1 = D_{12}(r, \theta) \tag{5.2}$$

Employing the rule of differentiation of composite functions (Sokolnikoff and Redheffer, 1958) the derivatives of $D_{11}(r, \theta)$ w.r.t. x_1 and $D_{12}(r, \theta)$ w.r.t. x_2 are

$$\begin{aligned} \frac{\partial D_{11}(r,\theta)}{\partial x_1} &= \cos(\theta)\frac{\partial D_{11}(r,\theta)}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial D_{11}(r,\theta)}{\partial \theta} \\ \frac{\partial D_{12}(r,\theta)}{\partial x_2} &= \sin(\theta)\frac{\partial D_{12}(r,\theta)}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial D_{12}(r,\theta)}{\partial \theta} \end{aligned} \tag{5.3}$$

Eq. (B.1) indicates that D_{11} and D_{12} are composite functions of constants, powers of $\cos(\theta)$ and $\sin(\theta)$, and powers of r. The functions forming D_{11} and D_{12} can all be represented by convergent power series and so D_{11} and D_{12} are analytic functions except at $r = 0$. Thus, by (5.3) $T_{11}^{(1)}$ and $T_{12}^{(1)}$ must be analytic functions except at $r = 0$ and so can be represented by potential functions of the complex variable

$$w = x_1 + \mathrm{i}x_2 \tag{5.4}$$

Therefore,

$$T_{11}^{(1)} = T_{11}^{(1)}(w) \text{ and } T_{12}^{(1)} = T_{12}^{(1)}(w) \tag{5.5}$$

Similar arguments may be used to demonstrate that the other stresses occurring in (3.5a) and (3.5b) may be represented as potential functions of the complex variable w. In addition, by (4.6) $T_{33}^{(p)}$ may also be represented as a potential function of the complex variable w.

Therefore,

$$T_{ij}^{(p)} = T_{ij}^{(p)}(w) \tag{5.6}$$

6. Magnitude of the in-plane stresses

Eq. (3.6) indicates the generalized plane strain stresses are linear combinations of the auxiliary generalized plane strain stresses. Therefore, by (5.6) it is possible to represent the generalized plane strain stresses as potential functions of the complex variable w

$$T_{ij} = \sum_{p=1}^{4} a_p T_{ij}^{(p)}(w) \tag{6.1}$$

Recall the equation for the stresses in Cartesian coordinates

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W}) \tag{3.4}$$

It can be shown by arguments similar to those preceding (5.4) that the terms in (3.4) not containing $T_{ij}(\mathbf{W})$ are analytic and may be represented by potential functions of the complex variable w. Therefore, when (6.1) is also taken into consideration, all the terms in (3.4) may be represented as potential functions of the complex variable w, then

$$S_{ij} = S_{ij}(w) \tag{6.2}$$

Recall the objective of this paper was to determine the magnitudes of S_{rr}, $S_{\theta\theta}$, and $S_{r\theta}$. A counter clockwise rotation about the x_3-axis will transform the stress tensor in Cartesian coordinates ($S_{\alpha\beta}$) back to cylindrical coordinates ($S'_{\alpha\beta}$).

$$S'_{ij} = Q_{im} Q_{jn} S_{mn} \tag{6.3}$$

Here

$$Q_{ij} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and θ is the cylindrical coordinate measured from the positive x_1-axis. Note

$$\begin{bmatrix} S'_{11} & S'_{12} & S'_{13} \\ S'_{21} & S'_{22} & S'_{23} \\ S'_{31} & S'_{32} & S'_{33} \end{bmatrix} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} \tag{6.4}$$

Consider $i = 1$, 2 and $j = 3$, then $Q_{ij} = 0$. Therefore, (6.3) may be written as

$$S'_{\alpha\beta} = Q_{\alpha\rho} Q_{\beta\gamma} S_{\rho\gamma} \tag{6.5}$$

Eq. (6.5) shows that the $S'_{\alpha\beta}$ are functions of only S_{11}, S_{22} and S_{12}. Consider the boundary condition on Π in cylindrical coordinates

$$s(\mathbf{u})' = S'_{ij} n_j = 0 \quad \text{on } \Pi \tag{6.6}$$

Recall on Π that $n_3 = 0$, therefore, substituting (6.5) into (6.6) results in

$$S'_{\alpha\beta} n_\beta = 0 \quad \text{on } \Pi \tag{6.7}$$

Recall from (6.4) that $S'_{11} = S_{rr}$, and $S'_{12} = S_{r\theta}$, then from (6.7)

$$S_{r\theta} = -\frac{n_1}{n_2} S_{rr} \quad \text{on } \Pi \tag{6.8}$$

Expanding (6.3) for S_{rr} results in

$$S_{rr} = C_\theta^2 S_{11} + 2 C_\theta S_\theta S_{12} + S_\theta^2 S_{22} \tag{6.9}$$

However, recall on Π that the boundary conditions for the stresses in Cartesian coordinates result in

$$S_{11} = \frac{-S_{12}\sin(\theta)}{\cos(\theta)} \quad S_{22} = \frac{-S_{12}\cos(\theta)}{\sin(\theta)} \quad \text{on } \Pi \tag{6.10}$$

Substituting (6.10) into (6.9) and simplifying results in

$$S_{rr} = -\cos(\theta)\sin(\theta)S_{12} + 2\cos(\theta)\sin(\theta)S_{12} - \cos(\theta)\sin(\theta)S_{12} = 0 \quad \text{on } \Pi \tag{6.11}$$

Substitute (6.11) into (6.8), then

$$S_{r\theta} = -\frac{n_1}{n_2}S_{rr} = 0 \quad \text{on } \Pi \tag{6.12}$$

Recall from (6.4) that $S'_{22} = S_{\theta\theta}$, then from (6.7) when taking into account (6.12)

$$S_{\theta\theta} = -\frac{n_1}{n_2}S_{r\theta} = 0 \quad \text{on } \Pi \tag{6.13}$$

Note from (6.2) and (6.3) that the $S'_{\alpha\beta}$ maybe represented as potential functions of the complex variable w, then from (6.11)–(6.13)

$$S_{rr}(w) = S_{\theta\theta}(w) = S_{r\theta}(w) = 0 \quad \text{on } \Pi \tag{6.14}$$

Recall from (5.3) that the $T^{(p)}_{ij}$ may not be analytic at $r = 0$. However, this is an isolated singular point and so may be excluded from the domain of analyticity by a circle of infinitely small radius. Thus, by Cauchy's integral formula, where from (6.14) $S'_{\alpha\beta}(\mathbf{u}) = S'_{\alpha\beta}(w) = 0$ on Γ and $w_o \neq 0$, then

$$S'_{\alpha\beta}(w_o) = \frac{1}{2\pi i}\int_{\Gamma} \frac{S'_{\alpha\beta}(w)}{w - w_o}\mathrm{d}w = 0 \tag{6.15}$$

However, since $w_o = 0$ is an isolated singularity, w_o may be made infinitely close to zero, and then from (6.15)

$$S'_{\alpha\beta}(w) = 0 \quad \text{in } \overline{\Sigma} \tag{6.16}$$

7. Conclusions

By Theorem 1 and Corollary 1 the auxiliary generalized plane strain stresses are only dependent on the cylindrical coordinate r, and so by (3.6) the generalized plane strain stresses depend on r alone. This fact will simplify the calculation of the constants a_p from the necessary conditions for a solution.

The generalized plane strain stresses were proven to be analytic functions that could be represented as potential functions of the complex variable w. In addition the S_{ij} stresses were recognized as composite functions, where the component functions were all analytic functions. The stresses in the plane of Σ were proven to equal zero on the lateral surface Π when considered in cylindrical coordinates, and this result was extended to the region $\overline{\Sigma}$. Thus, even when the constitutive equations in cylindrical coordinates are dependent on r, the stresses in the plane of a transverse cross-section are still equal to zero.

The results from this paper will be used to derive the three dimensional stress functions for a cylindrical section of a tree in the sequel, where the constitutive equations are linearly dependent on the cylindrical coordinate r.

Appendix A. Transformation equations

The transformation equations taking the elasticity coefficients in cylindrical coordinates (C'_{ijkl}) to Cartesian coordinates (C_{ijkl}) (Lyons et al., 2002).

$$
\begin{aligned}
C_{1111} &= C_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2222}\\
C_{2222} &= S_\theta^4 C'_{1111} + 2C_\theta^2 S_\theta^2 C'_{1122} + 4C_\theta^2 S_\theta^2 C'_{1212} + C_\theta^4 C'_{2222}\\
C_{3333} &= C'_{3333}\\
C_{2323} &= S_\theta^2 C'_{1313} + C_\theta^2 C'_{2323}\\
C_{1313} &= C_\theta^2 C'_{1313} + S_\theta^2 C'_{2323}\\
C_{1212} &= C_\theta^2 S_\theta^2 [C'_{1111} - 2C'_{1122} + C'_{2222} - 2C'_{1212}] + [C_\theta^4 + S_\theta^4] C'_{1212}\\
C_{1122} &= C_\theta^2 S_\theta^2 C'_{1111} + C_\theta^4 C'_{1122} - 4C_\theta^2 S_\theta^2 C'_{1212} + S_\theta^4 C'_{2211} + C_\theta^2 S_\theta^2 C'_{2222}\\
C_{1133} &= C_\theta^2 C'_{1133} + S_\theta^2 C'_{2233}\\
C_{1123} &= 0\\
C_{1113} &= 0\\
C_{1112} &= -C_\theta S_\theta [C_\theta^2 C'_{1111} - C_\theta^2 C'_{1122} - 2C_\theta^2 C'_{1212} + 2S_\theta^2 C'_{1212} + S_\theta^2 C'_{1122} - S_\theta^2 C'_{2222}]\\
C_{2233} &= S_\theta^2 C'_{1133} + C_\theta^2 C'_{2233}\\
C_{2223} &= 0\\
C_{2213} &= 0\\
C_{2212} &= -C_\theta S_\theta [S_\theta^2 C'_{1111} - S_\theta^2 C'_{1122} - 2S_\theta^2 C'_{1212} + 2C_\theta^2 C'_{1212} + C_\theta^2 C'_{1122} - C_\theta^2 C'_{2222}]\\
C_{3323} &= 0\\
C_{3313} &= 0\\
C_{3312} &= -C_\theta S_\theta [C'_{3311} - C'_{3322}]\\
C_{2313} &= -C_\theta S_\theta [C'_{1313} - C'_{2323}]\\
C_{2312} &= 0\\
C_{1312} &= 0
\end{aligned}
\tag{A.1}
$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

For the transformation equations taking the compliance coefficients in cylindrical coordinates (S'_{ijkl}) to Cartesian coordinates (S_{ijkl}), replace C'_{ijkl} with S'_{ijkl} and C_{ijkl} with S_{ijkl} in Eq. (A.1).

Appendix B. Transformed compliance coefficients

The non-zero transformation equations taking the compliance coefficients in Eq. (2.3) (S'_{ijkl}) to Cartesian coordinates (S_{ijkl}), when considering the simplifications from Eqs. (2.6)–(2.8) are as follows.

$$
\begin{aligned}
S_{1111} &= \lfloor C_\theta^4 + S_\theta^4 \rfloor \underline{S_{1111}} + r\lfloor C_\theta^4 \underline{M_{1111}} + S_\theta^4 \underline{M_{2222}} \rfloor + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
S_{1122} &= \underline{S_{1122}} \\
S_{1133} &= \underline{S_{1133}} + r[C_\theta^2 \underline{M_{1133}} + S_\theta^2 \underline{M_{2233}}] \\
S_{1112} &= -S_\theta C_\theta [C_\theta^2[\underline{S_{1111}} + r\underline{M_{1111}}] - C_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + S_\theta^2 \underline{S_{1122}} - S_\theta^2[\underline{S_{1111}} + r\underline{M_{2222}}]] \\
S_{2222} &= [C_\theta^4 + S_\theta^4] \underline{S_{1111}} + r[S_\theta^4 \underline{M_{1111}} + C_\theta^4 \underline{M_{2222}}] + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
S_{2233} &= \underline{S_{1133}} + r[S_\theta^2 \underline{M_{1133}} + C_\theta^2 \underline{M_{2233}}] \\
S_{2212} &= -S_\theta C_\theta [S_\theta^2[\underline{S_{1111}} + r\underline{M_{1111}}] - S_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + C_\theta^2 \underline{S_{1122}} - C_\theta^2[\underline{S_{1111}} + r\underline{M_{2222}}]] \\
S_{3333} &= \underline{S_{3333}} \\
S_{3312} &= -S_\theta C_\theta r[\underline{M_{1133}} - \underline{M_{2233}}] \\
S_{2323} &= \underline{S_{1313}} + r[S_\theta^2 \underline{M_{1313}} + C_\theta^2 \underline{M_{2323}}] \\
S_{2313} &= -S_\theta C_\theta r[\underline{M_{1313}} - \underline{M_{2323}}] \\
S_{1313} &= \underline{S_{1313}} + r[C_\theta^2 \underline{M_{1313}} + S_\theta^2 \underline{M_{2323}}] \\
S_{1212} &= C_\theta^2 S_\theta^2 [2\underline{S_{1111}} + r[\underline{M_{1111}} + \underline{M_{2222}}] - 2[\underline{S_{1122}} + \underline{S_{1212}}]] + [C_\theta^4 + S_\theta^4] \underline{S_{1212}}
\end{aligned}
\tag{B.1}
$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

Similar equations can be formed for the elastic coefficients by replacing the S_{ijkl} by C_{ijkl}, $\underline{S_{ijkl}}$ by $\underline{C_{ijkl}}$, and $\underline{M_{ijkl}}$ by $\underline{K_{ijkl}}$, in Eq. (B.1).

References

Iesan, D., 1987. Saint-Venant's Problem. Lecture Notes in Mathematics, no. 1279. Springer, New York, 161 p.

Lekhnitskii, S.G., 1981. In: Theory of Elasticity of an Anisotropic Body. MIR Publishers, Moscow, 430 p.

Lyons, C., Kevin, Guenther, R.B., Pyles, M.R., 2002. Elastic equations for a cylindrical section of a tree. International Journal of Solids and Structures 39, 4773–4786.

Sokolnikoff, I.S., Redheffer, R.M., 1958. In: Mathematics of Physics and Modern Engineering. McGraw-Hill Book Company, New York, 812 p.

US Department of Agriculture (USDA), 1974. Wood Handbook: Wood as an Engineering Material, Agriculture Handbook No. 72. Forest Products Laboratory, Forest Service, US Department of Agriculture.